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Macroeconomic Volatilities and Long-Run Risks of Asset Prices

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In this paper, motivated by existing and growing evidence on multiple macroeconomic volatilities, we extend the long-run risks model by allowing both a long- and a short-run volatility components in the evolution of economic fundamentals. With this extension, the new model not only is consistent with the volatility literature that the stock market is driven by two, rather than one, volatility factors, but also provides significant improvements in fitting various patterns, such as the size of market risk premium, the level of interest rate, degree of dividend yield predictability, and the term structure of variance risk premiums, of both the equity and option data.

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1. Introduction

In macroeconomics and finance literature, a central question is how macrofluctuations and market volatility affect the economy and asset prices. Since the early consumption-based asset pricing models of Lucas (1978) and Breeden (1979), the long-run risks model of Bansal and Yaron (2004; henceforth, BY) seems another major advance in this direction. In the BY model, persistent movement in volatility of aggregate consumption is introduced and shown to drive the conditional volatilities and risk premia of asset returns. There are subsequently many studies, such as Bansal et al. (2005; 2007b, c; 2009; 2012; 2014), Eraker (2008), Hansen et al. (2008), Pakoš (2008), Avramov and Hore (2009), Chen et al. (2009), Beeler and Campbell (2012), Constantinides and Ghosh (2011), Drechsler and Yaron (2011), Bansal and Shaliastovich (2013), Ferson et al. (2013), and Johnson and Lee (2014), that have shown that the long-run risks channels can successfully account for a number of features of the data, such as size of market risk premium, unconditional levels and volatilities of asset prices, predictability of return levels and volatilities by asset valuations, and a host of empirical facts in equity, bond, and option markets. However, the original BY model assumes a single variance state variable, which proportionally affects the short and long-run consumption innovations. An immediate implication of this assumption is that all the conditional volatilities of macro and financial market variables are driven by the same underlying consumption volatility.

In this paper, we introduce a new volatility factor into the fundamental BY model. This is motivated by recent growing evidence of multiple volatility factors in the aggregate economy. Alizadeh et al. (2002), Chernov et al. (2003), Chacko and Viceira (2005), and Adrian and Rosenberg (2008), among others, document multiple volatility factors. In studies on options, Christoffersen et al. (2008) and Lu and Zhu (2010), among others, find that two volatility factors are necessary in explaining option returns. Recently, Christoffersen et al. (2012, 2013) and Johnson (2012) demonstrate the importance of multiple volatility structure in the cross section of option prices. In terms of macro aggregates, Nakamura et al. (2012) identify multiple consumption volatility factors (global and local) from a large panel of consumption data for developed countries, and they find improved matching on predictability moments relative to the BY model with multiple volatilities. Decomposing the aggregate consumption into two components, Boguth and Kuehn (2013) estimate a two-volatility process and find volatility risk are important for the cross section of stock returns.

Theoretically, there are also various motivations on two-factor-volatility models. Bansal and Shaliastovich (2010) introduce "confidence risks" from an alternative channel to generate a second volatility factor from learning and fluctuations in investor confidence from the forecast data. Extending Drechsler and Yaron (2011), Branger et al. (2012) separate the processes for the jump intensity and the stochastic conditional variance, with the second volatility arising from the stochastic jump intensity. In contrast with these studies, our two-factor model is from a different motivation, and so the empirical applications are also different.

In our model, the two volatilities enter the model mainly from an investor's consumption channel. The first factor is to capture the time-varying variance of consumption growth, and the other is to reflect the time-varying uncertainty of expectation of consumption growth. The first factor is consistent with evidence in macroeconomics (see, e.g., Stock and Watson 2002). Bansal and Yaron (2004) pioneer the interpretation of the second, but they assume one common factor drives both the time-varying variance and the time-varying uncertainty of expectation of consumption growth. In contrast, we extend this to allow two factors to drive the two uncertainties. Our assumption of a two-volatility model for the consumption growth is broadly consistent with existing and growing evidence on multiple macroeconomic volatilities, especially the aforementioned recent studies.

There are three major empirical results with the two-volatility extension of the BY model. First, the data strongly support the two-volatility specification over the conventional one-volatility models in the literature based on the generalized method of moments (GMM) tests. Second, the introduction of the second volatility factor leads to a significant, first-order improvement in matching the standard features of the equity market data considered in the literature, such as the predictability of future cash flows, returns, and return volatilities by asset valuations. Third, the second volatility factor plays an important role in accounting for the more recent, option-based features of the markets data such as the variance risk premium and VIX term structure.

Methodologically, instead of calibrations, we provide perhaps one of the first empirical estimations of the long-run risks model. By casting our extension of the BY model in continuous time, we are able not only to avoid the occasional negative volatility problem in the discrete-time counterparts, but also to obtain approximate analytical solutions for many functions of economic interest, such as derivative prices, measures of volatility, and the slope coefficients of various predictive regressions. These provide qualitative insights in understanding long-run risks models as to where the one-factor BY model may need improvement and how the second volatility factor can help. In addition, they are valuable for empirical implementations. With the analytical solutions, it is straightforward to write out the model moment conditions, making it easy to apply GMM for estimation and tests.

The rest of this paper is organized as follows. Section 2 provides a short review of the BY model. Section 3 provides the new long-run risks model with both long-run and short-run volatility components and solves various functions of interest approximate analytically. Section 4 provides the empirical results. Section 5 discusses some of the open issues and future research, and §6 concludes.

2. A Short Review of the BY Model

In this section, we provide a short review of the BY model, which will be useful for understanding our extension and its comparison with other models.

The BY model assumes a representative investor who has Epstein–Zin–Weil recursive preferences (Epstein and Zin 1989, Weil 1989) and maximizes the lifetime utility,

$$V_t = [C_t^{(1-\gamma)/\theta} + \beta(E_t[V_{t+1}^{1-\gamma}])^{1/\theta}]^{\theta/(1-\gamma)}, \tag{1}$$

where C_t is consumption at time t, γ is the coefficient of risk aversion, ψ is the intertemporal elasticity of substitution (IES), and $\theta = (1-\gamma)/(1-1/\psi)$. When $\theta = 1$, i.e., $\gamma = 1/\psi$, the recursive utility becomes additive power utility; when $\theta < 1$, i.e., $\gamma > 1/\psi$, as demonstrated by Epstein and Zin (1989), the investor prefers early resolution of uncertainty, which, intuitively, is due to the time aggregator being a convex function with power $1/\theta > 1$. As illustrated by Bansal and Yaron (2004), it is necessary to impose $\gamma > 1$ and $\psi > 1$ to account for the high stock market risk premium in the long-run risks model.

The investor makes his optimal portfolio decision under the discrete-time processes for consumption and dividends, the economic fundamentals, as follows:

$$\begin{aligned} \log(C_{t+1}/C_t) &= \mu + X_t + \sigma_t \eta_{t+1}, \\ X_{t+1} &= \alpha X_t + \varphi_x \sigma_t e_{t+1}, \\ \sigma_{t+1}^2 &= \sigma^2 + \kappa(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}, \\ \log(D_{t+1}/D_t) &= \mu_d + \phi X_t + \varphi_d \sigma_t u_{t+1}, \end{aligned} \tag{2}$$

where X_t is the long-run risk that affects both consumption and dividend growth, and is persistent with autoregression (AR) coefficient α and volatility $\varphi_x \sigma_t$; the variance process σ_t^2 is also an AR process with coefficient κ; η_{t+1}, e_{t+1}, w_{t+1}, and u_{t+1} are independent shocks drawn from the standard normal distribution; the parameter ϕ is the dividend growth leverage ratio; and φ_x and φ_d are volatility leverage ratios for long-run risks and dividend growth, respectively.

The intuition behind the model is that X_t captures the long-run growth prospects of the economy. Shocks in both long-run X_t and short-run η_{t+1} drive the consumption growth and asset prices. With $\theta < 1$, the fear of adverse long-run growth requires a high risk premium as compensation. Since the long- and short-run shocks in dividend growth and asset returns can be very volatile, the BY model can successfully explain, among other stylized facts, the equity risk premium, the risk-free rate, and the volatility of the market return.

However, there are some strong implications from the original BY model that are inconsistent with the data. Empirically, the price–dividend ratio has little power in predicting consumption growth, but the model implies the opposite. Bansal et al. (2012; henceforth, BKY) address this issue by increasing the persistence of the volatility to make it a more important factor, so the relative importance of the long-run consumption risk factor in the price–dividend ratio can be reduced, and so can its predictability on consumption growth. However, this persistent volatility factor implies much greater predictive power of stock price–dividend ratio for future stock return volatility than is found in the data, as pointed out by Beeler and Campbell's (2012) extensive study. In addition, the increase in the persistence of the volatility entails a small (virtually zero) variance risk premium that is inconsistent with the data. Drechsler and Yaron (2011) allow jumps in the volatility process that are capable of explaining the large negative variance risk premium, but their extension does not resolve the predictability problem. Because multifactor stochastic volatility models are necessary in capturing volatility risk premium and the volatility term structure dynamics (see, e.g., Christoffersen et al. 2008, Lu and Zhu 2010, Zhou and Zhu 2011), we adopt this framework in this paper to allow two volatility factors in the long-run risks model.

3. The New Long-Run Risks Model

In this section, we first motivate our dynamic processes for the state variables in the new long-run risks model, and then solve the model in terms of the state variables. Subsequently, we provide approximate analytical solutions to functions of interest: the consumption–wealth ratio, market prices of risks, price–dividend ratio, and market return volatility. Then we derive analytical results for the predictability regression coefficients of the excess return, consumption and dividend growth rate, and their volatilities, as well as the variance risk premium.

3.1. The Model and Solution

Our model extends the BY and BKY models in the continuous-time framework with two volatility factors. Parallel to the discrete-time model (1), we consider the following model for the consumption and dividend processes and their related variables:

$$
\begin{aligned}
\frac{dC_t}{C_t} &= (\mu + X_t)\,dt + \sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}\,dZ_{1t},\\
dX_t &= -\alpha X_t\,dt + \varphi_x\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}\,dZ_{2t},\\
\frac{dD_t}{D_t} &= (\mu_d + \phi X_t)\,dt + \varphi_d\sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}\,dB_t\\
&\quad + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}\,dZ_{1t}\\
&\quad + \sigma_{dx}\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}\,dZ_{2t}\\
&\quad + \sigma_{dv}\sqrt{V_{1t}}\,dw_{1t} + \sigma_{dv2}\sqrt{V_{2t}}\,dw_{2t},\\
dV_{1t} &= \kappa_1(\bar{V}_1 - V_{1t})\,dt + \sigma_1\sqrt{V_{1t}}\,dw_{1t},\\
dV_{2t} &= \kappa_2(\bar{V}_2 - V_{2t})\,dt + \sigma_2\sqrt{V_{2t}}\,dw_{2t}, \quad 0<\kappa_1<\kappa_2,
\end{aligned}
\tag{3}
$$

where dZ_{1t}, dZ_{2t}, dB_t, dw_{1t}, and dw_{2t} are independent Brownian motions. This model specification, which sets a convenient framework for our GMM estimation and model comparison study in later sections, is the most general one for two-factor models that allow for analytical solutions, and nests both continuous-time BY and BKY models.[1] When $\delta_x = 1$, $\delta_c = 1$, $\delta_d = 1$, and $\sigma_{dc} = \sigma_{dx} = \sigma_{dv} = \sigma_{dv2} = 0$, the model reduces to the continuous-time BY model. When $\delta_x = 1$, $\delta_c = 1$, $\delta_d = 1$, and $\sigma_{dc} = \sigma_{dv2} = 0$, it becomes the continuous-time BKY model. Furthermore, if we set $\delta_x = 1$, $\delta_c = 1$, $\delta_d = 1$, and $\sigma_{dv2} = 0$, we get the most general one-factor model.

The key feature of the new model is that the consumption growth has two volatility factors, whose motivation is discussed in the introduction. In this setup, the total variance level is $V_{1t}\delta_c + V_{2t}(1-\delta_c)$, a convex combination of a long- and a short-run variance, V_{1t} and V_{2t}. The same variance decomposition, adjusted by the volatility leverage factor φ_x, is also applied to the long-run risk X_t. The dividend growth process is treated similarly, except that it allows for various covariations with C_t and X_t, as in the BKY model. The two components of volatility, as derived from the model, will enter into the stock price volatility, which helps to match the market variance premium and the volatility predictability with the data. Furthermore, the long- and short-run variances follow two independent standard square-root Heston (1993) processes, which avoid the negative variance problem of the discrete-time Gaussian specification of the BY and BKY models.

To solve the equilibrium prices and other quantities of interest, following the BY model, we use the Epstein–Zin–Weil preference, but in continuous time. Based on

[1] In the original BY and BKY models, variance is modelled as Gaussian process as opposed to square-root process in this paper. Hence we refer to the BY and BKY models as long-run risks models with one-factor square-root variance process.

Duffie and Epstein (1992) and analogous to (1), we can define the intertemporal value function recursively by

$$J_t = E_t\left[\int_t^T f(C_s, J_s)\,ds\right]. \tag{4}$$

Thus, the representative investor's objective is to choose consumption to optimize the value function; that is,

$$J_t = \max_{\{C_s\}} E_t\left[\int_t^T f(C_s, J_s)\,ds\right], \tag{5}$$

where $f(C, J)$ is a *normalized aggregator* related to current consumption C_t and continuation value function J_t, and is given by

$$f(C_t, J_t) = \frac{\beta}{1-1/\psi}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{1/(1-\gamma)}}\right)^{1-1/\psi} - 1\right], \tag{6}$$

with β the rate of time preference, $\gamma > 0$ the relative risk aversion, and $\psi > 0$ the IES. Theoretically, the aggregator in (6) should be an increasing function of the value function J (see, e.g., Skiadas 2009). This places joint restrictions on γ and ψ such that $\theta \geq 1$ or $\theta < 0$, where $\theta = (1-\gamma)/(1-1/\psi)$. If $\theta = 1$, as shown by Duffie and Epstein (1992), we obtain the standard additive expected utility of constant relative risk aversion. In our model setting, the value function satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\max_{\{C\}}\{f(C, J) + \mathscr{A}^c J\} = 0, \tag{7}$$

where $\mathscr{A}^c$ is the infinitesimal generator associated with vector process $(C_t, X_t, V_{1t}, V_{2t})$ defined in Equation (3). The solutions are obtained by conjecturing a function form of value function as

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t})\frac{W_t^{1-\gamma}}{1-\gamma}, \tag{8}$$

and using the log-linear approximation, which Campbell (1993) develops in discrete time and Chacko and Viceira (2005) use first in continuous time. The details are provided in the appendix.

3.2. Consumption–Wealth Ratio

The consumption–wealth ratio

$$\frac{C_t}{W_t} = \beta^{\psi}\exp\{(A_{0a} + A_{1a}X_t + A_{2a}V_{1t} + A_{3a}V_{2t})\} \tag{9}$$

is log linear in the state variables and has similar functional form as in the BY model. In particular,

$$A_{1a} = -\frac{1-1/\psi}{g_1 + \alpha}, \tag{10}$$

which is exactly the same as the continuous analogue of BY's $-A_1$.[2] Hence, the same interpretation applies that, when $\psi > 1$ ($\psi < 1$), $A_{1a} < 0$ ($A_{1a} > 0$), which means that a rise in expected consumption growth lowers (increases) the consumption–wealth ratio, and the substitution (income) effect dominates. In addition, the consumption–wealth ratio is more sensitive to the expected growth rate when the persistence of expected growth shocks, measured by $1/\alpha$, increases.

However, there are now two volatilities in Equation (9). This is expected since they are in the basic dynamics equations. Because of their symmetric formulations entered into the consumption and dividend dynamics, the two volatility components impact on the ratio in the same way, with proportional coefficients A_{2a} and A_{3a} depending on the volatility parameters via similar functional forms. When $\psi > 1$, both A_{2a} and A_{3a} are positive. The same intuition of the original BY model about volatility also holds here. For example, a rise in either of the volatilities will make consumption more volatile, which lowers asset valuations and increases the risk premia on all assets. In addition, an increase in the persistence of volatility shocks, that is, a decrease in either κ_1 or κ_2, will magnify the effects of volatility shocks on valuation ratios, since the investor would perceive changes in economic uncertainty as being long lasting.

Empirically, the aggregate wealth, which consists of both asset wealth and human capital, is not directly observable. However, Jagannathan and Wang (1996) and Lettau and Ludvigson (2001), among others, argue that it may be expressed in observable variables of consumption, asset wealth, and labor income under various assumptions. In this paper, we focus on the stock price–dividend ratio and its predictability, which are empirically observable.

3.3. Asset Prices

In this subsection, we present the analytical results for the risk-free rate, market prices of risks, and stock price–dividend ratio whose derivations are in the appendix. The risk-free rate is given by

$$r_f = r_0 + r_1 X_t + r_2 V_{1t} + r_3 V_{2t}, \tag{11}$$

with parameters given in (35). Note that $r_1 = 1/\psi > 0$, which implies that the risk-free rate rises with higher expectation of consumption growth.

The state price density (or pricing kernel) can be expressed as

$$\frac{d\pi_t}{\pi_t} = -(r_f\,dt + \lambda_1 dZ_{1t} + \lambda_2 dZ_{2t} + \lambda_3 dw_{1t} + \lambda_4 dw_{2t}), \tag{12}$$

[2] Note that Bansal and Yaron (2004) use the ratio of wealth to consumption, but we use the ratio of consumption to wealth. Hence our A_{ia}'s have the opposite sign of theirs. The same applies to the price–dividend ratio below.

where λ_i's ($i = 1, 2, 3, 4$) are the market prices of risks compensating for short-run consumption risk, long-run consumption risk, and two-volatility risks, respectively. The maximal Sharpe ratio is defined as the conditional volatility of the pricing kernel, $\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2}$. When $\gamma = 1/\psi$, the case for standard models, such as the Breeden (1979) consumption-based capital asset pricing model (CCAPM), all the risk premia other than that of the short-run consumption risk become zero, and hence it will not be possible for the standard models to match the market risk premium. However, the long-run risks models with $\gamma > 1/\psi$, including the BY and BKY models and our new model, can produce the positive market prices for the long-run risks and negative market prices for variance risks, both of which, combined with the stock market's positive beta for long-run risks and negative beta for volatility risks, contribute positively to solving the equity risk premium puzzle. Hence, the relative magnitudes of λ_i's quantify the different contributions to the maximal Sharpe ratio by various risk factors, and will be examined empirically later with comparison to the BY model.

The price–dividend ratio is

$$\frac{D_t}{P_t} = \exp\{(A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})\}. \quad (13)$$

The price process is obtained by applying Ito's lemma to Equation (13):

$$\frac{dP_t}{P_t} = [c_3 + c_4X_t + c_5V_{1t} + c_6V_{2t}]dt + \sqrt{c_1V_{1t} + c_2V_{2t}}\, dZ_t,$$

where c_i's ($i = 1$ to 6) are constants given in (38) and (39). The term dZ_t is a new Brownian motion defined accordingly, and hence the variance of the price process is

$$V_t = c_1V_{1t} + c_2V_{2t}. \quad (14)$$

3.4. Predictability of Excess Returns, Consumption, and Dividends

To examine predictability of the variables, we, following Beeler and Campbell (2012), consider the following three K-period regressions:

$$(r_{t+j} - r_{f,t+j}) + \cdots + (r_{t+j+K} - r_{f,t+j+K}) = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt}, \quad (15)$$

$$\Delta c_{t+j} + \cdots + \Delta c_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt}, \quad (16)$$

$$\Delta d_{t+j} + \cdots + \Delta d_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt}, \quad (17)$$

where r and r_f are the stock market rate of return and risk-free rate, respectively, and c_t and d_t are logarithms of consumption and dividends. To explain the observed regression patterns, our idea is to derive the regression slope coefficients as functions of the model parameters that can be chosen in such a way as to make the model-implied regression slope coefficients match closely with those of the data.

We provide the formulas for $K = 1$ only for notational simplicity, and the general case is a straightforward extension. As $K = 1$, the regressors of the above three regressions all have the generic functional form of

$$dY_t = [a_0 + a_1X_t + a_2V_{1t} + a_3V_{2t}]\, dt + \sqrt{b_1V_{1t} + b_2V_{2t}}\, dZ_t, \quad (18)$$

where dY_t corresponds to excess return $d\ln P_t + D_t/P_t - r_f dt$, consumption growth $d\ln C_t$, and dividend growth $d\ln D_t$, respectively. The parameter a_i as well as the regression slope coefficient are then analytically obtained in Equation (40), which shows that with two volatility components rather than one, the covariance function in the numerator of β depends on three rather than two factors, precisely how the volatility components and the model parameters contribute to the degree of predictability.

3.5. Predictability of Volatility: Excess Returns, Consumption, and Dividends

In this subsection, we present the predictive coefficients of volatility by excess returns, consumption growth, and dividend growth, which are not matched well in the BY and BKY models, as demonstrated in Beeler and Campbell (2012). In this paper, we use the same volatility measure as in Beeler and Campbell (2012). There are two steps in computing this measure. First, we run an AR(1) regression of each variable of interest y_{t+1} as

$$y_{t+1} = b_0^{\text{vol}} + b_1^{\text{vol}} y_t + u_{t+1}, \quad (19)$$

where y_{t+1} is the excess return or consumption growth or dividend growth. Second, the K-period realized volatility is defined as the sum of the absolute values of the residuals,

$$\text{Vol}_{t,\,t+K-1} = \sum_{k=0}^{K-1} |u_{t+k}| \quad (20)$$

over K periods, with K, as before, the horizon of interest.

Then, the predictability of volatility is examined from the regression of the log of K-period realized volatility on the logarithm of price–dividend ratio,

$$\ln[\text{Vol}_{t+1,\,t+K}] = \alpha + \beta_{\text{vol}}(p_t - d_t) + \xi_t. \quad (21)$$

The derivations are given in the appendix.

3.6. Variance Risk Premium

In this subsection, we follow Drechsler and Yaron (2011), among others, to define the variance risk premium (VRP) as the difference between the objective or physical expectation and the risk-neutral expectation of the variance of aggregate stock market return over a period τ_0. The risk-neutral expectation of variance is also known as the variance swap (VS) rate, which can be replicated by model-free method utilizing options on all strike prices. Variance swap contracts are popular over-the-counter volatility derivatives betting on realized variance (RV) against the predetermined VS rate (see Egloff et al. 2009). The VIX index, which has been published by the Chicago Board Options Exchange (CBOE) since 2003, is the square root of the VS rate on S&P 500 index with 30-day maturity (for details, see Chicago Board Options Exchange 2009, Zhang and Zhu 2006). In fact, as Lu and Zhu (2010) and many others show, the VIX is almost always bigger than the realized volatility because investors with more risk aversion are willing to pay a positive risk premium for long stock market volatility. For that reason, VIX is referred to as the market's fear gauge by financial commentators.

Following Zhang and Zhu (2006), we first define the time t expected future realized variance over period τ_0 under the physical measure, which is denoted as RV_t. Using the market variance process defined in Equation (14), we have[3]

$$\mathrm{RV}_t = \sum_{i=1}^{2} c_i (A_i^P + B_i^P V_{it}). \tag{22}$$

Similarly, the VS rate, defined as the time t expected future realized variance over time period τ_0 under the risk-neutral probability, is given by

$$\mathrm{VS}_t = \sum_{i=1}^{2} c_i (A_i^Q + B_i^Q V_{it}). \tag{23}$$

The parameters A_i^P, B_i^P,A_i^Q, and B_i^Q are defined in the appendix.

Hence the VRP, defined as the difference between RV of Equation (47) and the VS rate of Equation (48), can be expressed as

$$\mathrm{VRP} \equiv \mathrm{RV}_t - \mathrm{VS}_t = \sum_{i=1}^{2} c_i [(A_i^P - A_i^Q) + (B_i^P - B_i^Q) V_{it}], \tag{24}$$

where the coefficients are analytically determined by the model parameters.

In our model, the variance risk premium is negative, indicating that investors regard increases in market volatility as unfavorable shocks to the investment opportunity. In comparison with the BY and BKY models, the market variance risk premium is determined by both the long- and short-run volatilities. As will be clear later, because of the rich dynamics of these two components, the associated parameters can be chosen in such a way that the model can explain the market variance risk premium along with other facts, whereas the previous models fail to do so.

[3] The RV is annualized, and so is the variance swap rate as well as the VRP. The VRP data provided in Tables 1, 2, and 7 are annualized with a scaling factor 10,000/12.

4. Empirical Results

In this section, we first describe the data and estimation procedure for our empirical study. We then present the empirical results and model comparison with BY and BKY models. Finally, we examine how the two-volatility-factor model explains the predictability issues raised by Beeler and Campbell (2012) as well as the large and negative variance risk premium of the data.

4.1. Data Description

The stock market and macroeconomic data from 1930 and 2008 are from Beeler and Campbell (2012).[4] The stock index returns are monthly. The risk-free rates are the 30-day returns on the Treasury bills. The consumption data include total nominal nondurables and services consumption, which are deflated both by the rate of population growth and by inflation. The population data are the year-end values from the Census Bureau. These values are used for annual data and the fourth quarter of quarterly data. Other quarterly population values are interpolated assuming a constant geometric growth rate within a year. Stock returns, dividend growth, and consumption are all deflated with the consumer price index (CPI). For yearly inflation, the rate of inflation is the log of the ratio of the CPI in the last month of the current year to the CPI in the last month of the previous year.

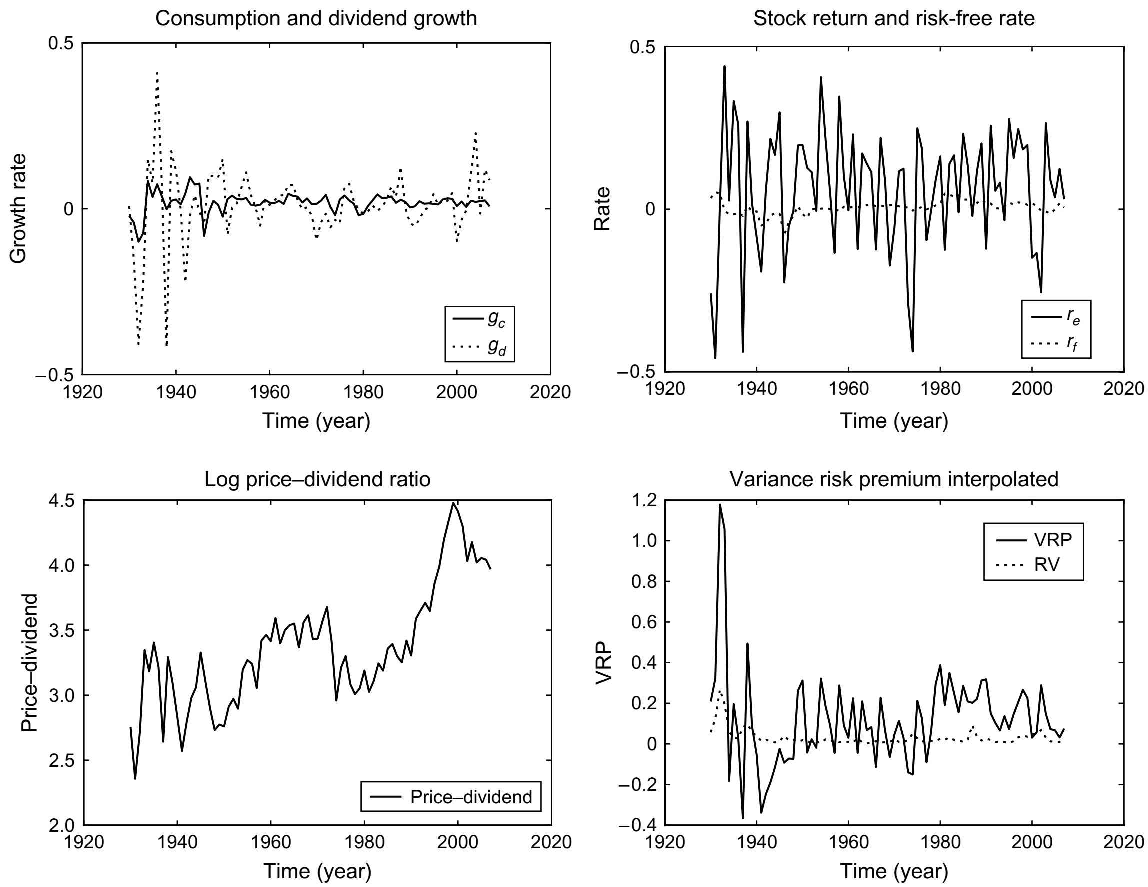
We compute the model-free VRP from CBOE's VIX series from 1990 to 2008 and the realized monthly variance based on the S&P 500 index. Due to the lack of VIX, and hence VRP, data before 1990, we run the following quarterly regression of VRP with data from 1990 to 2008, with $R^2 = 0.11$:

$$\begin{aligned}\mathrm{VRP}/100 = {} & -0.122 + 3.89 g_c - 0.094 g_d - 0.837 r_e \\ & + 0.018\mathrm{pd} - 4.56 r_f - 1.94\mathrm{RV}/100 + \epsilon,\end{aligned} \tag{25}$$

where g_c is the consumption growth, g_d is the dividend growth, r_e is the excess stock return, pd is the log price–dividend ratio, r_f is the risk-free rate, and RV is the realized stock variance. Then we use the fitted regression to extrapolate the VRP data from 1930 to 1990. The extrapolation approach is often used in the bond literature to fill in missing data. The extrapolation preserves the mean value of VRP well. Over the entire time period from 1930 to 2008, the mean of the VRP is

[4] We are grateful to Jason Beeler for providing us the data.

Figure 1 **VRP Regression**



−13.11, fairly comparable to −15.85, the mean of the observed data after 1990. In addition, its covariance with stock return is −0.40 and −0.36, respectively, for the extended sample and original sample. Overall, the fitted data are comparable for the original and extended samples. Figure 1 plots all the data of both the regressand and the regressors. The absolute value of VRP peaks during the Great Depression periods.

Following many studies, such as BKY, we are interested in the long-horizon implications of the long-run risks models, and hence use the annual data to estimate the model, because not all data are available over 1930–1990 periods with higher frequency. Table 1 provides the summary statistics of the annualized data. The consumption growth has large fat tails as measured by the high kurtosis value. Note that the interest rate is in real terms, so it has a minimum value of −7.98%.

4.2. GMM Tests

We use GMM to estimate and test the model. The base case pricing restrictions include the mean value, standard deviation, and first-order autocorrelation coefficients of the log consumption growth, log dividend growth, real risk-free rate, excess stock return, and log stock price–dividend ratio. These 15 base case moments, which are presented in the left panel of Table 2, have been extensively studied in the long-run risks model literature, e.g., Bansal and Yaron (2004), Bansal et al. (2007b, c), and Beeler and Campbell (2012).

Besides the base case moments, we pay special attention to the restrictions on six predictability regression coefficients given in Equations (40) and (44). Given the unconditional moments of the base case, these predictability coefficients will be sensitive to the additional state variable of the second volatility factor. Finally, we include two additional restrictions on variance risk premium, the mean and standard deviation. These eight additional moments are shown in the right panel of Table 2.

Table 1 **Summary Statistics**

	g_c	g_d	p_d	r_e	r_f	VRP
Mean	1.95	1.02	3.31	6.20	0.99	13.11
Std. dev.	2.16	10.69	0.46	18.34	4.28	23.95
Skewness	−1.16	−0.92	0.47	−0.68	−0.76	1.74
Kurtosis	6.99	8.77	2.93	3.32	5.53	9.85
Min	−9.95	−41.81	2.36	−45.91	−7.98	−36.60
Max	9.50	40.93	4.48	43.97	5.18	117.83

Note. This table provides the mean, standard deviation, skewness, kurtosis, minimum, and maximum of the annual data from 1930 to 2008.

Table 2 GMM Moments

	Data	BY	BKY	New
$E(\Delta c)$	1.95	1.79	1.82	1.73
$\sigma(\Delta c)$	2.16	2.92	2.96	2.21
AC1(Δc)	0.44	0.51	0.44	0.45
$E(\Delta d)$	1.02	1.66	1.85	1.32
$\sigma(\Delta d)$	10.69	11.57	16.42	11.25
AC1(Δd)	0.14	0.40	0.29	0.32
$E(r_e)$	6.20	6.62	6.58	6.34
$\sigma(r_e)$	18.34	16.88	21.35	19.16
AC1(r_e)	0.04	0.03	0.02	0.01
$E(r_f)$	0.99	2.56	0.99	1.15
$\sigma(r_f)$	4.28	1.30	1.28	1.84
AC1(r_f)	0.59	0.85	0.86	0.65
$E(p-d)$	3.31	3.00	3.04	2.56
$\sigma(p-d)$	0.46	0.16	0.26	0.26
AC1$(p-d)$	0.88	0.77	0.95	0.85

	Data	BY	BKY	New
$\beta(r_e)$	−0.059	−0.007	−0.078	−0.073
$\beta(\Delta c)$	0.012	0.114	0.022	0.032
$\beta(\Delta d)$	0.064	0.343	0.054	0.174
$\beta_{\text{vol}}(r_e)$	−0.081	−0.123	−1.315	−0.167
$\beta_{\text{vol}}(\Delta c)$	−0.481	−0.128	−1.420	−0.336
$\beta_{\text{vol}}(\Delta d)$	−0.530	−0.146	−1.483	−0.561
VRP	−13.11	−0.005	−0.010	−5.79
σ(VRP)	23.95	0.000	0.000	29.71

Notes. This table presents the 23 target moments of the data. The fitted moments of the BY and BKY models are taken from Beeler and Campbell (2012). The data are annual from 1930 to 2008.

In our GMM test, we minimize a weighted sum of squared differences of the 23 target moments between the model derived and the market implied. Instead of using the standard optimal weighting matrix, we use a diagonal one with weights adjusting the moments to the same order of magnitude. This technique is used by Zhou (1994) to isolate parameters for making parameter estimation easy and stable. Then, the GMM test statistic can be constructed to account for the special weighting matrix (see the online appendix associated with this paper for more details, available at http://apps.olin.wustl.edu/faculty/zhou/Online_Appendix_Feb2014.pdf).

4.3. Moment Matching and Model Comparison

In this subsection, we present the GMM estimation and test results.

Since the two mean levels of the two volatility factors, $\bar{V}_1$ and $\bar{V}_2$, are not identified independently, we set $\bar{V}_1 = \bar{V}_2$. Hence, there are in our two-factor model a total of 21 parameters, $\theta = (\mu, \alpha, \varphi_x, \delta_c, \delta_x, \mu_d, \phi, \varphi_d, \delta_d, \sigma_{dc}, \sigma_{dx}, \sigma_{dv}, \sigma_{dv2}, \bar{V}_1 = \bar{V}_2, \sigma_1, \kappa_1, \sigma_2, \kappa_2, \beta, \gamma, \psi)$.

Table 3 provides the estimation results under the heading "New." For comparison, we also provide the corresponding values of the continuous-time version of the BY and BKY models based on their calibrations.

There are a few notables. First, all the parameters governing the consumption growth, μ, α, and φ_x, are virtually the same across the models.[5] Second, the parameters of the dividend growth process are, however, different between their models and ours. In our model, the dividend growth leverage ratio ϕ is 4.53, whereas the values for BY and BKY are 3 and 2.5. Intuitively, increasing ϕ will increase A_{1m}, the price–dividend ratio sensitivity to the long-run risk component. This increase is required by the need to match

Table 3 Long-Run Risks Parameters

	Preference parameters		
	γ	ψ	β
BY	10	1.5	0.024
BKY	10	1.5	0.03
New	10.100	1.350	0.009
Error	2.274	0.985	0.115

	Consumption growth dynamics				
	μ	α	φ_x	δ_c	δ_x
BY	0.018	0.256	0.528	1	1
BKY	0.018	0.3	0.456	1	1
New	0.018	0.256	0.528	0.915	0.176
Error	0.002	0.115	0.040	0.172	0.227

	Dividend growth dynamics							
	μ_d	ϕ	φ_d	δ_d	σ_{dc}	σ_{dx}	σ_{dv}	σ_{dv2}
BY	0.018	3	4.5	1	0	0	0	0
BKY	0.018	2.5	5.96	1	2.6	0	0	0
New	0.019	4.532	2.121	1.000	0.076	−3.033	2.000	−2.993
Error	0.002	0.015	0.007	0.034	0.003	0.007	0.005	0.007

	Volatility parameters					
	Factor 1			Factor 2		
	$\bar{V}_1$	σ_1	κ_1	$\bar{V}_2$	σ_2	κ_2
BY	0.027^2	0.0035	0.156			
BKY	0.025^2	0.0027	0.015			
New	0.024^2	0.0029	0.038	0.024^2	0.676	10.745
Error	0.0003	0.0331	0.563	0.0003	0.116	0.007

Notes. This table reports the parameters for three long-run risks models: the Bansal and Yaron (2004), Bansal et al. (2012), and new two-factor models; γ is the risk aversion parameter, ψ is the IES parameter, and β is the discount rate. Other panels provide parameters governing the consumption, dividend, and volatility dynamics in Equation (3).

[5] Note that the long-run risk volatility multiple φ_x is 0.044 and 0.038 for BY and BKY models, respectively. We have multiplied them by 12 here because X_t is the annualized expected consumption growth in our model, whereas it is monthly in their model.

the predictability regression parameters. The dividend volatility leverage ratio is $\varphi_d = 4.5$ for BY, $\sqrt{\varphi_d^2 + \sigma_{dc}^2} = 6.5$ for BKY, and $\sqrt{\varphi_d^2 + \sigma_{dc}^2 + \sigma_{dx}^2 + \sigma_{dv}^2 + \sigma_{dv2}^2} = 5.16$ for the new model, which is between the BY and BKY models. Third, the substantial difference occurs for the volatility process, as expected. Whereas the volatility of the BKY model appears too persistent, with a half life of around 50 years, and much more so than that of the BY model of 4.5 years, our new model has a long-run component with a half life of around 20 years and a short-run one of around one month, which are consistent with aforementioned literature on stochastic volatility. In addition, note that once we incorporate the two components in the volatility process, the long-run (short-run) component should have a half life longer (shorter) than that implied from the one-component model. This is indeed the case when we compare our model with the BY calibration.

To understand the models further, it will be useful and informative to assess the moments and their sample estimates, as did Beeler and Campbell (2012). Table 2 presents the 23 moments that are used in our GMM estimates. We provide the results given by BY, BKY, our new model, and the data (with yearly time interval from 1930 to 2006). The left panel provides the 15 base case asset pricing moments discussed. As with the BY and BKY models, these moments are well matched by our model. In particular, our estimated preference parameters, $\gamma = 10.1$ and $\psi = 1.35$, are close to the BY calibration of $\gamma = 10$ and $\psi = 1.5$. In addition, our model produces all the stylized facts on the market, with an equity risk premium of 6.34% and a risk-free rate of 1.15%, close to the market data of 6.20% and 0.99%, respectively. Furthermore, Table 4 presents the comparison between our new model and the BY model on the conditional volatility of the pricing kernel (the maximal Sharpe ratio) and the relative contributions of different economic shocks. The result of the BY model is from Bansal and Yaron (2004). The maximal Sharpe ratio in our model is 0.60, compared with 0.73 of the BY model, the difference of which is mainly due to the modelling differences, such as the continuous-time versus discrete-time model, and the square-root process versus the Gaussian process for variance. Furthermore, the contribution from the fluctuating economic uncertainty in our new model comes from two volatility factors, in contrast to BY's one volatility factor. Despite the differences, the relative contributions from the different risks to the pricing kernel are roughly the same. For example, the contribution of the long-run risks is 49% in the BY model, and 47% in the new model. In addition, the combined contribution from the two volatility factors in the new model is 36%, compared with the 39% of the one volatility factor in the BY model. Similar to the BY model, the independent consumption shocks contribute only 15% to the total variance of the pricing kernel, and the maximal Sharpe ratio with only independent and identically distributed consumption growth rate is only 0.23. In conclusion, our model matches well the basic asset pricing moments studied in the BY model, and the main difference between the models comes from the different structure of the volatility factors.

Table 4 **Decomposing the Variance of the Pricing Kernel**

	Relative variance of shocks				
	Volatility of pricing kernel	Independent consumption (%)	Expected growth rate (%)	Fluctuating economic uncertainty (%): Factor 1	Fluctuating economic uncertainty (%): Factor 2
New	0.60	15	49	25	11
BY	0.73	14	47	39	

Notes. This table presents the volatility of the pricing kernel (maximal Sharpe ratio) and the relative contributions of different shocks to the variance of the pricing kernel, for both our new model and the BY model. The result of our new model is based on the GMM estimation, and that for the BY model is from Bansal and Yaron (2004). The contribution from the fluctuating economic uncertainty for our new model comes from two volatility factors, in contrast to BY's one volatility factor.

The major differences among the models occur in the right panel of Table 2, where $\beta(r_e)$, $\beta(\Delta c)$ and $\beta(\Delta d)$ are the regression coefficients of the market excess return, consumption, and dividend growth on the price–dividend ratio in a one-year horizon. For the market excess return, the data imply $\beta(r_e) = -0.059$. Whereas our model matches well with a beta of $\beta(r_e) = -0.073$, the BY model, with a beta of $\beta(r_e) = -0.007$, does not match this important predictability. In contrast, the BKY model improves over the BY model substantially in this aspect, matching well with a beta of $\beta(r_e) = -0.078$. Overall, the BKY model performs very well in explaining the predictability of excess returns, consumption, and dividends, and so does our new model.

However, the good performance of the BKY model comes at a high cost of matching the predictive regression coefficients of the volatility regressions of excess return, consumption, and dividend growth on the price–dividend ratio, $\beta_{\mathrm{vol}}(r_e)$, $\beta_{\mathrm{vol}}(\Delta c)$, and $\beta_{\mathrm{vol}}(\Delta d)$. For $\beta_{\mathrm{vol}}(r_e)$, the BKY value is -1.315, which is more than 10 times greater in magnitude than that of the data, -0.081. For both $\beta_{\mathrm{vol}}(\Delta c)$ and $\beta_{\mathrm{vol}}(\Delta d)$, the values of the BKY model are about three times greater in magnitude than those of the data too. On the other hand, the BY model provides volatility predictability 10 times smaller than that of BKY model, and fails to match the data as well. In contrast, our new model matches well with the data for all the three volatility regression coefficients. As explained later in §4.5, the key reason is that both BY and BKY models rely on a

Table 5 Predictability for Different Horizons

Rate regression		Data		BY		BKY		Two-factor	
	Periods	β	R^2	β	R^2	β	R^2	β	R^2
r_e	1Y	−0.059	0.022	−0.007	0.000	−0.078	0.009	−0.073	0.012
r_e	3Y	−0.229	0.143	−0.026	0.000	−0.226	0.026	−0.192	0.083
r_e	5Y	−0.421	0.278	−0.039	0.000	−0.368	0.041	−0.297	0.198
g_c	1Y	0.012	0.068	0.114	0.390	0.022	0.037	0.039	0.210
g_c	3Y	0.010	0.013	0.286	0.435	0.052	0.042	0.031	0.044
g_c	5Y	−0.001	0.000	0.350	0.373	0.069	0.036	0.025	0.017
g_d	1Y	0.064	0.074	0.343	0.228	0.054	0.008	0.174	0.173
g_d	3Y	0.076	0.034	0.860	0.288	0.133	0.011	0.138	0.036
g_d	5Y	0.051	0.013	1.171	0.265	0.176	0.010	0.112	0.014

Volatility regression		Data		BY		BKY		Two-factor	
	Periods	β_{vol}	R^2	β_{vol}	R^2	β_{vol}	R^2	β_{vol}	R^2
r_e	1Y	−0.081	0.001	−0.123	0.000	−1.315	0.085	−0.167	0.004
r_e	3Y	−0.059	0.003	−0.115	0.001	−1.268	0.273	−0.096	0.004
r_e	5Y	−0.017	0.000	−0.113	0.002	−1.336	0.364	−0.080	0.004
g_c	1Y	−0.481	0.035	−0.128	0.000	−1.420	0.095	−0.546	0.045
g_c	3Y	−0.491	0.122	−0.122	0.001	−1.382	0.290	−0.527	0.129
g_c	5Y	−0.564	0.235	−0.113	0.002	−1.336	0.375	−0.509	0.206
g_d	1Y	−0.530	0.035	−0.146	0.000	0.054	0.102	−0.317	0.013
g_d	3Y	−0.478	0.070	−0.120	0.002	0.133	0.305	−0.267	0.024
g_d	5Y	−0.496	0.084	−0.103	0.003	0.176	0.393	−0.250	0.034

Notes. This table reports the regression coefficients and R^2 values for the rate regressions and the volatility regressions in the upper and lower panels, respectively. The predictive variable is the log price–dividend ratio, and r_e, g_c, and g_d are stock returns, consumption growth, and dividend growth, respectively. Y, year.

one-factor volatility model, which makes the matching difficult for $\beta_{\text{vol}}(r_e)$, $\beta_{\text{vol}}(\Delta c)$, and $\beta_{\text{vol}}(\Delta d)$.

To further assess the predictability, Table 5 reports both the regression slopes and the R^2 values for various horizons, one, three, and five years. The table shows that our two-factor model can generate the "term structure" of predictability as demonstrated in the data. For example, in the data, the term structure of R^2 increases with the horizon for stock returns (from 0.022 to 0.278), whereas it decreases for both consumption and dividend growth rates. Our model matches the pattern well, but the BY model, with only one-factor volatility, can only produce a flat term structure in which the short-run predictability is closely related to the long-run predictability. The BKY model has performance similar to that of the BY model.

Furthermore, we examine the last two target moments in Table 2, i.e., the mean and standard deviation of the VRP. Whereas the data[6] show a large negative VRP at −13.11 with standard deviation of 23.95, both the BY and the BKY models imply a VRP of only −0.005 and −0.010, which are too small in magnitude to explain the observed variance risk premium. In contrast, our new model implies a VRP of −5.79, which is very close to the empirical mean level. Moreover, the standard deviation of the VRP in our model is 29.71, which also matches well with the data. The J test of the model is the standardized sum of squared moment matching errors; J should be χ^2-distributed with degree of 2. Our model has $J = 1.8$, corresponding to a p-value of $p = 0.40$, which means we cannot reject the model at all conventional confidence levels. Specifically, the BKY model can be obtained through restrictions, i.e.,

$$\bar{V}_2 = \sigma_2 = \kappa_2 = \sigma_{dv2} = 0$$

and

$$\delta_c = \delta_x = \delta_d = 1.$$

The BY model can be obtained by additional restrictions of

$$\sigma_{dc} = \sigma_{dx} = \sigma_{dv} = 0.$$

We use the same covariance matrix for the matching moments as the unrestricted two-factor model and compute $J(\text{BKY}) - J(\text{New}) \sim \chi^2(7)$ and $J_T(\text{BY}) - J_T(\text{New}) \sim \chi^2(10)$ using the moments given by BKY

[6] According to market convention, the VRP data presented here are converted to monthly VRP through a factor of 1/12, and multiplied by 10,000.

and BY, which shows that $J_T(\text{BKY}) - J_T(\text{New}) = 1{,}133$ and $J_T(\text{BY}) - J_T(\text{New}) = 688$, a very large increase in the errors for matching moments, corresponding to a p-value very close to zero. This means that our new model outperforms the BY and the BKY models substantially from a statistical point of view as well.

4.4. Return Variance Decomposition

To obtain further intuition of the model, we examine its implication on return variance decomposition, which was first proposed by Campbell and Shiller (1988) and inspired a large volume of literature across many disciplines including finance, accounting, and macroeconomics (for a review, see Chen and Zhao 2009). In particular, using aggregate stock market data from 1927 to 1988, Campbell (1993) shows that about one-third of the variance of unexpected returns is attributed to the cash-flow risk, one-third to the discount-rate risk, and one-third to the covariance between these two. Debate on which one is more important in driving stock market return variation, discount-rate risk, or cash-flow risk, is highly inconclusive because of model uncertainty in return predictability. Using data from 1953 to 2001 with different predictive variables, Chen and Zhao (2009) find that discount-rate risk accounts for anywhere from 10% to more than 80% of return variance.

The long-run risks model, successful in explaining the level and volatility of stock market return, attributes almost all unexpected variation of the stock return to changing cash flow by emphasizing on cash-flow risk. In contrast, the habit formation model by Campbell and Cochrane (1999) emphasizes discount-rate risk by changing investors' risk attitude. We show that by extending the one-volatility BY model to the two-volatility model, we provide enough flexibility for the model to generate more variation due to discount-rate risk.

We apply the variance decomposition of Campbell (1993) with our model estimation. Table 6 demonstrates the results of our new model in comparison with the BKY model.[7] Note that in our estimation, BKY attribute 99.5%, 8.2%, and −7.6% of total variation to cash-flow risk, discount-rate risk, and covariance between these two, respectively, whereas Avramov and Cederburg's (2012) estimations are 106%, 4.4%, and −11.2%, respectively. The difference is due to the fact that we cast the BY and BKY models in continuous time with a square-root volatility process, in contrast to their discrete-time models with a Gaussian volatility process. Both cases show that the BKY model attributes too small a portion of stock return variance to discount-rate than aforementioned studies. On the other hand, our new two-factor model attributes 25.8% to discount-rate news, which is in line with the empirical results of Campbell (1993). The ability of our two-volatility model to generate more discount-rate variation is attributed to the additional short-run volatility factor, which has much higher volatility of volatility (vol of vol) and hence higher volatility of the discount rate. It turns out that this short-run volatility factor is also very important to explain volatility predictability and variance risk premium, as demonstrated next.

Table 6 Variance Decomposition: Contribution to Total Variance

	BKY model	New model
Discount-rate risk	0.082	0.258
Cash-flow risk	0.995	0.661
Covariance	−0.076	0.081
Total variance	1.000	1.000

Notes. This table reports the attribution of return variance to discount-rate shocks, cash-flow shocks, and covariance between the two shocks. The decomposition is performed on the BKY model and the new two-factor model.

4.5. Volatility Predictability

In this subsection, we examine further the statistical properties of the volatility regression coefficients. Table 2 shows that for the one-year horizon, the regression slope coefficients β_{vol} for volatilities of excess return, consumption, and dividends are, respectively, −0.123, −0.128, and −0.146 in the BY model, and −1.315, −1.420, and −1.483 in the BKY model, both of which show little difference across the three regressions. The results are consistent with our theoretical analysis. Equation (43) shows that the volatility regression coefficients should be the same for all three variables based on a one-factor model. Indeed, the three regressors in regression Equation (21) can all be written as

$$\ln \int_0^\tau \sqrt{V_t}\,|dZ_t| \approx Const + \frac{1}{2\tau}\int_0^\tau \frac{V_t}{\bar{V}}\,dt, \qquad (26)$$

where V_t corresponds generically to volatilities of excess return, consumption, and dividend growth. Note that in the one-factor volatility model, the three volatilities will be the same up to a scaling factor to the single volatility component; hence the regressors are all the same because $V_t/\bar{V}$ will be invariant to a scaling factor. Consequently, β_{vol}'s will be the same for the three variables, and this holds for different horizons. Hence, it is critical to use a two-factor volatility model to explain the varying degrees of volatility predictability.

Note that the volatility predictability in the BKY model is tenfold larger than that in the BY model, because of a much more persistent volatility component in the BKY model. In our proposed two-factor model, the volatility regression slope coefficients match remarkably well with those from the data. While the data imply significantly varying beta values of −0.081,

[7] Here we only compare with BKY calibration because BKY calibration matches predictability better than BY calibration.

−0.481, and −0.530 for excess return, consumption growth, and dividend growth, respectively, so does the model with −0.167, −0.336, and −0.561.

In short, the additional volatility factor is instrumental to explain the differences in volatility predictability across excess returns, consumption, and dividends. As it turns out, it is also fundamentally important in explaining the market variance premium, as discussed in §4.6.

4.6. Variance Risk Premium

As mentioned above, one of the difficulties of the BY and BKY models is to explain the market variance risk premium, which is large and negative. In this subsection, we examine in detail why the BY and BKY models fail to match the market VRP. Consider the following BY/BKY one-volatility-factor process, which is cast in our continuous setting:

$$dV_t = \kappa(\bar{V} - V_t)\, dt + \sigma\sqrt{V_t}\, dw_t.$$

The model-implied VRP is $-\nu V_t$, with the coefficient ν given by

$$\nu = \frac{1-\gamma\psi}{1-\gamma} A_2 \sigma^2. \qquad (27)$$

For a persistent process with κ very small, given the observable level of unconditional vol of vol, $\sigma^2\bar{V}/(2\kappa)$, which is inversely proportional to κ, the parameter σ^2 and hence the VRP coefficient ν in Equation (27) cannot be large enough to match the market data. In BKY calibration, with a value of $\kappa = 0.012$ and an annual return volatility of 20%, the VRP in absolute value will be bounded by

$$|\text{VRP}_{\text{BKY}}| < 0.012 \times (20\%^2/12) \times 10{,}000 = 0.4,$$

which is much smaller than a value of 13.11 from our data,[8] as reported in Table 7. With an additional volatility component that is much less persistent than the volatility processes in both BY and BKY, our new model can produce the desired VRP mostly contributed by this short-run volatility, which is consistent with evidence from the volatility derivatives market (e.g., Egloff et al. 2009, Lu and Zhu 2010). In addition, our new model matches the autoregression coefficient AR1 (with one month lag) of the VRP well. Whereas the data imply an AR1 value of 0.54, our model yields a value of 0.47. This is also consistent with that of Bollerslev et al. (2009). In contrast, both BY and BKY models imply a too-large value of 0.99 for AR1, a result of their single-factor structure of the volatility process that is highly persistent.

[8] Note that the VRP estimated using Drechsler and Yaron (2011) data is −12.67. The difference between their data and ours is due to the difference in sampling period.

Table 7 Variance Risk Premium

	Data (DY)	DY	Data (ours)	BY	BKY	New
VRP	−12.67	−7.57	−13.11	−0.005	−0.010	−5.79
Std.	14.38	10.65	23.95	0.000	0.000	29.71
AR1	0.54	N/A	0.15	0.99	0.99	0.47

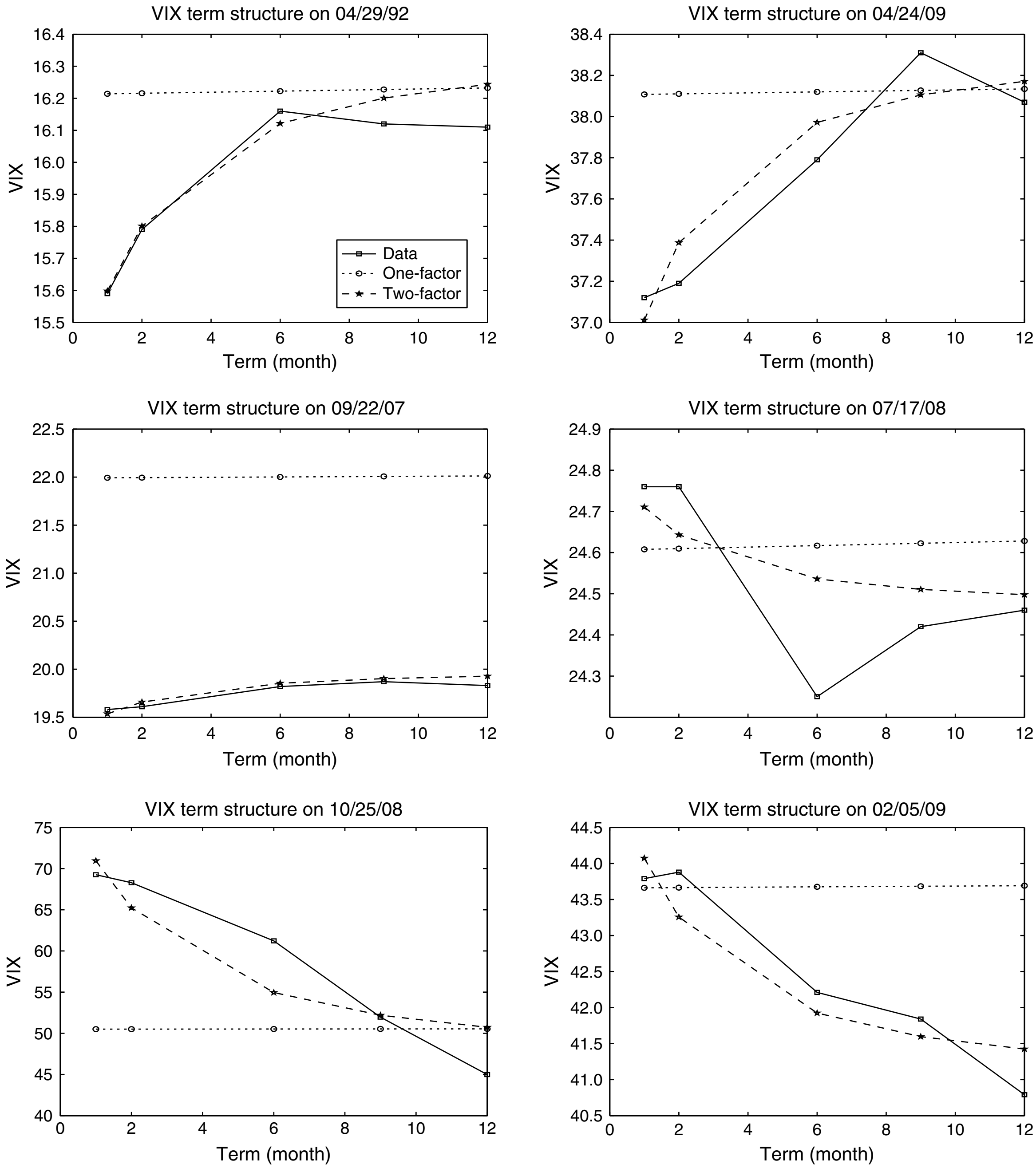
Notes. This table reports variance risk premiums (VRP), the standard deviation (Std.) of VRP, and the monthly autoregression coefficient (AR1) for the market data, the Drechsler and Yaron (2011) (DY) model, the BY model, the BKY model, and our new models. The results of the first two columns are from Drechsler and Yaron (2011). The third column is the data computed from our data set. All the values are monthly and multiplied by 10,000.

As mentioned earlier, a jump-diffusion model such as that by Drechsler and Yaron (2011) is an alternative to explain the negative and large market VRP in the long-run risks model. Table 7, citing their results, shows that their model indeed explains well the magnitude of the VRP. Similar models are also proposed by Bansal et al. (2007a), Eraker (2008), Eraker and Shaliastovich (2008), and Bollerslev et al. (2012), among others. However, as evident from our earlier analysis, it is very difficult for the Drechsler and Yaron (2011) model to explain the volatility predictability cross sectionally because it has only one state variable. In addition, studies in the volatility literature (see, e.g., Christoffersen et al. 2008, Lu and Zhu 2010) show that the two-factor volatility model is generally preferred in explaining the large negative VRP as well as variance term structure. Hence, our extension of the BY and BKY models by adding another volatility factor seems to offer a promising route for future applications and further extension of the long-run risks models.

4.7. VIX Term Structure

In this subsection, we provide further evidence to support the two-factor volatility model. VIX term structure is perhaps the most direct evidence to show the two-factor volatility processes. The VIX term structure data are constructed by CBOE using the same methodology for VIX, but with different maturities. The data go back to January 1992, with contract maturity ranging from a few days to a few years. We construct a constant maturity term structure with maturities of 1, 2, 6, 9, and 12 months. We then use a Kalman filter to back out the latent volatility state variables V_1 and V_2 using the parameters of one- and two-factor models estimated using GMM. Then we construct the VIX term structure implied by the models for each day. We compute the mean squared errors (MSEs) for the five contracts with 1, 2, 6, 9, and 12 month maturities, defined as the average of squared pricing errors of the models. The MSE for the two-factor model is 0.72 with standard error 0.03, whereas that for the one-factor model is 3.39 with standard error 0.15; hence the difference is highly significant. The out-of-sample (OOS) results are also of interest. The OOS MSE for the one-factor model is 3.90,

Figure 2 **VIX Term Structure: One-Factor Model vs. Two-Factor Model**



Notes. This figure provides the VIX term structure based on one- and two-factor volatility models. The upper, middle, and lower panels show three scenarios of the possible shapes of the term structure, i.e., upward sloping, flat, and downward sloping. In each graph, the fitted term structures by one- and two-factor models are represented by dotted and dashed lines, respectively, and the data are represented by a solid line.

whereas that for the two-factor model is 1.78, which improves the one-factor model by over 50%.

To further appreciate the ability of the two-factor model for capturing the term structure of variance, we present in Figure 2 the VIX term structure based on the one- and two-factor volatility models. The upper, middle, and lower panels show three scenarios of the possible shapes of the term structure, i.e., upward sloping, flat, and downward sloping. The upward-sloped term structure implies the market expectation of the future volatility will increase, which usually occurs in normal times. The financial crisis period is featured by a downward-sloped term structure due to a dramatic increase in short-term volatility, as demonstrated in the lower panel of the figure. Generally, the one-factor model can only generate a flat term structure, whereas the two-factor model can generate both upward and downward shapes. To see this further,

Table 8 VIX Term Structure Slope

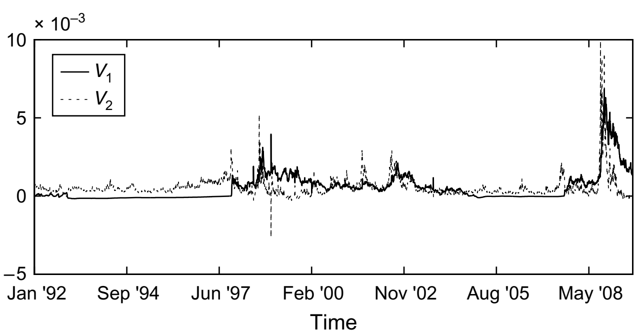
	Data	One-factor	Two-factor
Mean	0.19	0.02	0.23
Std.	3.52	0.00	2.54
Skew.	−2.29	3.66	−2.83
Kurt.	11.38	16.89	14.02
Min.	−34.85	0.02	−26.24
Max.	23.29	0.04	4.82

Note. This table reports the statistics of the VIX term structure slope, which is defined as the difference between the 12 month and 1 month contracts, from data and the one-factor and two-factor volatility models.

Table 8 demonstrates the statistics of the VIX term structure slope, which is defined as the difference between the 12 month and 1 month contracts, from the data and the one-factor and two-factor volatility models. It shows that the two-factor model matches the data very well, with a mean slope coefficient of 0.23 versus 0.19 for the data, and standard deviation 2.54 versus 3.52. On the other hand, the one-factor model cannot generate the patterns of the data because it captures only the level component. The explanation is similar to that for the interest rate term structure; that is, the one-factor model captures the level component, whereas the two-factor model can capture both the level and slope factors. Johnson (2012) applies principal component analysis to the VIX term structure and finds that the first component accounts for 95%, whereas the second component accounts for 4%, of the total movement of the curve.

Finally, it is of interest to see how the variance factors implied from the two-factor volatility model vary over time. Figure 3 plots their time series based on VIX term structure data from CBOE from January 1992 to September 2009. The V_1 is the long-run volatility factor with high persistence, and the V_2 is the short-run volatility. From 1992 through 1997, the long-run factor is extremely quiet, whereas short-run factor is generally higher and more volatile. During this period, the VIX index is low (average at 14%) and the term structure is generally flat. Between 1997 and 2003, both volatility factors are higher and more volatile, which might be driven by the Asian financial crisis, Russian default, and Internet bubble burst. During this period, the VIX index is higher (average at 25%) and the slope is also more volatile, sometimes positively sloped, sometimes negatively sloped. Between 2003 and 2007, the factors are more stable, and the levels are generally lower. The VIX index is back to a low level (average at 14%) again, with term structure generally flat. From 2008 through 2009, the factors become much higher again and swing fast, which seems to be driven by the 2008 financial crisis. During the crisis, the VIX index shoots up to above 50% and peaks at around 80%.

Figure 3 Latent Variance Factors



Notes. This figure shows the two variance factors implied from the two-factor volatility model using VIX term structure data from CBOE from January 1992 to September 2009. The V_1 is the long-run volatility factor with high persistence, and the V_2 is the short-run volatility.

5. Conclusion

Motivated by existing and growing evidence on multiple macroeconomic volatilities, this paper extends the one-factor Bansal and Yaron (2004) model with an additional volatility. Our extension is strongly supported by the data over the conventional one-volatility models in the literature. The introduction of the second volatility factor leads to a significant, first-order improvement in matching the standard features of the equity market data considered in the literature, such as return predictability and volatilities. Moreover, the second volatility factor plays an important role in accounting for the option markets data such as the variance risk premium and VIX term structure. However, the cross-section impact of the proposed factor and the pricing power in other asset markets are unknown and could serve as interesting topics for future research.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2014.1962.

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Appendix

In this appendix, we provide condensed proofs of the key results in the paper. Proofs of more results are available in the online appendix.

Model Solution and Asset Pricing Results

We conjecture a solution for J of Equation (7) as the following form:

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t}) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (28)$$

and use standard log-linear approximation, which Campbell (1993) developed in discrete time and Chacko and Viceira (2005) used first in continuous time (an accuracy assessment is provided at the end of this appendix). Specifically, let g_1 be the long-term mean of the consumption–wealth ratio,

$$g_1 = \exp(E[c_t - \omega_t]), \quad (29)$$

where the lowercase variables are the log variables.[9] With the standard log-linear approximation we have

$$\frac{C_t}{W_t} = \exp(c_t - \omega_t) \approx g_1 - g_1 \log g_1 + g_1 \log(C_t/W_t). \quad (30)$$

A key step is to derive consumption–wealth ratio explicitly in terms of state variables to substitute out the wealth in the value function. The first-order condition

$$f_C = J_W$$

leads to the consumption–wealth ratio as

$$\frac{C_t}{W_t} = \beta^{\psi} \exp(A_{0a} + A_{1a} X_t + A_{2a} V_{1t} + A_{3a} V_{2t}),$$

where $A_{ia} = A_i((1-\psi)/(1-\gamma))$ for $i = 0, 1, 2, 3$. Substituting out the wealth in J conjectured in Equation (8), the HJB equation can be solved by giving A_i's. The solution is approximate in general and exact when $\psi = 1$. In particular, we have

$$A_1 = \frac{1-\gamma}{(g_1+\alpha)\psi},$$

which is negative when $\gamma > 1$. In addition, A_2 and A_3 are both positive, which means that a rise in long-run consumption growth expectation increases the value function, whereas a rise in consumption volatility lowers the value function.

Define a state price process or pricing kernel π_t for any security with dividend process D_t and price process P_t as

$$P_t = \frac{1}{\pi_t} E_t\left[\int_t^{\infty} \pi_s D_s \, ds\right]. \quad (31)$$

In particular, for a risk-free asset with risk-free rate r_f, we have

$$-r_f \, dt = E_t\left[\frac{d\pi_t}{\pi_t}\right].$$

[9] It can be solved endogenously once the model parameters are known. The details are available upon request.

The Euler equation can be expressed in a differential form

$$E_t\left(\frac{dP_t}{P_t}\right) + \frac{D_t}{P_t} dt = r_f \, dt - E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right]. \quad (32)$$

Duffie and Epstein (1992) identify a state price process for the above defined recursive utility as

$$\pi_t = \exp\left[\int_0^t f_J(C_s, J_s)\, ds\right] f_C(C_t, J_t). \quad (33)$$

The wealth process is defined as the present value of consumption stream as

$$W_t = \frac{1}{\pi_t} E_t\left[\int_t^{\infty} \pi_s C_s \, ds\right]. \quad (34)$$

In our long-run risks model with two volatility factors, the pricing kernel is projected onto the risk space spanned by $\{Z_{1t}, Z_{2t}, w_{1t}, w_{2t}\}$ in Equation (3), and can be expressed as Equation (12), which can be derived by applying Ito's lemma to Equation (33). The risk-free rate is given in Equation (11), with the parameters

$$\begin{aligned}
r_0 &= -\left(\xi_1 + (\kappa_1 A_2 \bar{V}_1 + \kappa_2 A_3 \bar{V}_2)\frac{1-\gamma\psi}{1-\gamma} - \gamma\mu\right),\\
r_1 &= \frac{1}{\psi},\\
r_2 &= (g_1+\kappa_1) A_2 \tfrac{1-\gamma\psi}{1-\gamma} - \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2 (A_1^2 \varphi_x^2 \delta_x + A_2^2 \sigma_1^2) - \frac{1}{2}\gamma(\gamma+1)\delta_c,\\
r_3 &= (g_1+\kappa_2) A_3 \tfrac{1-\gamma\psi}{1-\gamma} - \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2 [A_1^2 \varphi_x^2 (1-\delta_x) + A_3^2 \sigma_2^2] - \frac{1}{2}\gamma(\gamma+1)(1-\delta_c).
\end{aligned} \quad (35)$$

The market prices of risks are

$$\begin{aligned}
\lambda_1 &= \gamma\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)},\\
\lambda_2 &= -\frac{1-\gamma\psi}{1-\gamma} A_1 \varphi_x \sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)},\\
\lambda_3 &= -\frac{1-\gamma\psi}{1-\gamma} A_2 \sigma_1 \sqrt{V_{1t}},\\
\lambda_4 &= -\frac{1-\gamma\psi}{1-\gamma} A_3 \sigma_2 \sqrt{V_{2t}}.
\end{aligned} \quad (36)$$

Now we derive price–dividend ratio from the market portfolio return defined as

$$r_{m,t} dt \equiv E_t\left(\frac{dP_t}{P_t}\right) + \frac{D_t}{P_t} dt = r_f \, dt - E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right], \quad (37)$$

where $r_{m,t}$ is the continuous compound market portfolio return. The price process is derived by Ito's lemma and given in Equation (13) with parameters

$$\begin{aligned}
c_1 &= \varphi_d^2 \delta_d + \sigma_{dc}^2 \delta_c + (\sigma_{dx} - A_{1m}\varphi_x)^2 \delta_x + (\sigma_{dv} - A_{2m}\sigma_1)^2,\\
c_2 &= \varphi_d^2 (1-\delta_d) + \sigma_{dc}^2 (1-\delta_c) + (\sigma_{dx} - A_{1m}\varphi_x)^2 (1-\delta_x) + (\sigma_{dv2} - A_{3m}\sigma_2)^2,
\end{aligned} \quad (38)$$

and the parameters for the drift term are

$$c_4 = \phi + \alpha A_{1m},$$
$$c_5 = \left(\tfrac{1}{2}A_{1m}^2\varphi_x^2\delta_x + \tfrac{1}{2}A_{2m}\sigma_1^2 + A_{2m}\kappa_1 - A_{1m}\sigma_{dx}\varphi_x\delta_x - A_{2m}\sigma_1\sigma_{dv}\right), \quad (39)$$
$$c_6 = \left(\tfrac{1}{2}A_{1m}^2\varphi_x^2(1-\delta_x) + \tfrac{1}{2}A_{3m}\sigma_2^2 + A_{3m}\kappa_2 - A_{1m}\sigma_{dx}\varphi_x(1-\delta_x) - A_{3m}\sigma_2\sigma_{dv2}\right). \quad \text{Q.E.D.}$$

Predictability of Variables

The predictability regression coefficients defined in Equations (15)–(17) are

$$\beta = \frac{\mathrm{Cov}(\Delta_\tau Y, p-d)}{\mathrm{Var}(p-d)}, \quad (40)$$

with $\Delta_\tau Y \equiv Y_{t+\tau} - Y_t$, and

$$\mathrm{Cov}(\Delta_\tau Y, p-d) = -\left[a_1 A_{1m}\frac{\sigma_x^2}{2\alpha^2}(1-e^{-\alpha\tau}) + a_2 A_{2m}\frac{\sigma_1^2\bar{V}_1}{2\kappa_1^2}(1-e^{-\kappa_1\tau}) + a_3 A_{3m}\frac{\sigma_2^2\bar{V}_2}{2\kappa_2^2}(1-e^{-\kappa_2\tau})\right],$$

with

$$\mathrm{Var}(p-d) = A_{1m}^2\frac{\sigma_x^2}{2\alpha} + A_{2m}^2\frac{\sigma_1^2\bar{V}_1}{2\kappa_1} + A_{3m}^2\frac{\sigma_2^2\bar{V}_2}{2\kappa_2}. \quad (41)$$

Q.E.D.

Predictability of Volatilities

First, we note that the innovation process u_{t+1} in the AR(1) process (19) is a discrete-time version of dZ_t in Equation (18). Thus, in the continuous limit, the discrete realized volatility defined in Equation (20) can be written as

$$\mathrm{Vol}_{t,\,t+\tau} = \int_0^\tau \sqrt{V_t}\,|dZ_t| = \frac{2}{\sqrt{2\pi}}\int_0^\tau \sqrt{V_t}\,dt, \quad (42)$$

where we have used the relation

$$|dZ_t| = \frac{2}{\sqrt{2\pi}}\,dt$$

for a standard Brownian motion Z_t. Then, a key step in approximating the log of τ period realized volatility in Equation (42) is to use the following approximate equality:

$$\ln\int_0^\tau \sqrt{V_t}\,|dZ_t| \approx Const + \frac{1}{2\tau}\int_0^\tau \frac{V_t}{\bar{V}}\,dt, \quad (43)$$

where $\bar{V}$ is the unconditional mean of V_t, and τ the horizon of interest similar to K. Then, we can obtain approximately the volatility predictive slope as

$$\beta_{\mathrm{vol}} = \frac{\mathrm{Cov}(\ln[\mathrm{Vol}_{t+1,\,t+K}], p-d)}{\mathrm{Var}(p-d)}, \quad (44)$$

where

$$\mathrm{Cov}(\Delta_\tau y, p-d) = -\frac{1}{2\tau\bar{V}}\left[b_1 A_{2m}\frac{\sigma_1^2\bar{V}_1}{2\kappa_1^2}(1-e^{-\kappa_1\tau}) + b_2 A_{3m}\frac{\sigma_2^2\bar{V}_2}{2\kappa_2^2}(1-e^{-\kappa_2\tau})\right],$$

with $\mathrm{Var}(p-d)$ as given by Equation (41), and b_1 and b_2 given by $b_1 = c_1$ and $b_2 = c_2$ in the case of excess return volatility; $b_1 = \delta_c$, $b_2 = 1-\delta_c$ in the case of consumption growth volatility; and

$$b_1 = \varphi_d^2\delta_d + \sigma_{dc}^2\delta_c + \sigma_{dx}^2\delta_x + \sigma_{dv}^2,$$
$$b_2 = \varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + \sigma_{dx}^2(1-\delta_x) + \sigma_{dv2}^2$$

in the final case of dividend growth volatility. Equation (44) shows nicely how the volatilities are predicted by the price–dividend ratio. Q.E.D.

Variance Risk Premium

We first derive the realized variance as

$$RV_t = \frac{1}{\tau_0}E_t^P\left[\int_t^{t+\tau_0} V_s\,ds\right] = E_t^P\left[\int_t^{t+\tau_0}\frac{1}{\tau_0}\sum_{i=1}^2 c_i V_{is}\,ds\right] = \sum_{i=1}^2 c_i\int_t^{t+\tau_0} ds\frac{1}{\tau_0}E_t^P[V_{is}], \quad (45)$$

where $E^P[\cdot]$ denotes the expectation under physical probability. Using the volatility factor dynamics defined in Equation (3), we have

$$E_t^P[V_{is}] = \bar{V}_i + (V_{it} - \bar{V}_i)e^{-\kappa_i(s-t)}, \quad (46)$$

for $i = 1, 2$. Substituting Equation (46) into Equation (45), we have

$$\mathrm{RV}_t = \sum_{i=1}^2 c_i(A_i^P + B_i^P V_{it}), \quad (47)$$

where A_i^P and B_i^P $(i = 1, 2)$ are constants given by

$$A_i^P = \bar{V}_i\left[1 - \frac{1-e^{-\kappa_i\tau_0}}{\kappa_i\tau_0}\right], \quad B_i^P = \frac{1-e^{-\kappa_i\tau_0}}{\kappa_i\tau_0}.$$

Then we derive the time t expected future realized variance over time period τ_0 under the risk-neutral probability. The market prices of risk for V_{1t} and V_{2t} are λ_3 and λ_4 of Equation (36); hence the associated risk premia are $\nu_1 V_{1t}$ and $\nu_2 V_{2t}$, which are proportional to V_{1t} and V_{2t}, and the risk-neutral processes are also square-root processes. The variance swap rate is given by

$$\mathrm{VS}_t = \sum_{i=1}^2 c_i(A_i^Q + B_i^Q V_{it}), \quad (48)$$

with A_i^Q and B_i^Q given as

$$A_i^Q = \frac{\kappa_i\bar{V}_i}{\kappa_i^Q}\left[1 - \frac{1-e^{-\kappa_i^Q\tau_0}}{\kappa_i^Q\tau_0}\right], \quad B_i^Q = \frac{1-e^{-\kappa_i^Q\tau_0}}{\kappa_i^Q\tau_0}. \quad (49)$$

Q.E.D.

Accuracy of the Log-linear Approximation

To assess the accuracy of the log-linear approximation solution (28) for parameter values of our interest, we consider three aspects. First, following Chacko and Viceira (2005), we know that the accuracy depends on small variations of the standard deviation of the log consumption–wealth ratio around its unconditional mean. In our case, they are less than 1.8%. Second, the additional volatility factor contributes less than 2% to the total standard deviation. This implies that the approximation error should be roughly the same as that for a one-factor model. Third, we find a similar nontrivial and tractable version of our two-factor model and verify that the errors are indeed small. The tractable model is

$$\frac{dC_t}{C_t} = \mu\, dt + \sqrt{V_t}\, dZ_t, \quad dV_t = \kappa(\bar{V} - V_t)\, dt + \sigma\sqrt{V_t}\, dw_t.$$

The value function can be written as

$$J(V_t, C_t) = e^{\mathcal{G}(V_t)} \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{50}$$

It can be shown that the solution for $\mathcal{G}(V)$ follows an ordinary differential equation as

$$\left[\frac{1}{\epsilon}(\beta - \beta e^{\epsilon\mathcal{G}(V)}) + (1-\gamma)\left(\mu - \frac{\gamma}{2}V\right)\right] + \kappa(\bar{V} - V)\frac{d\mathcal{G}(V)}{dV} + \frac{1}{2}\sigma^2 V\left(\frac{d\mathcal{G}(V)}{dV}\right)^2 + \frac{1}{2}\sigma^2 V\frac{d^2\mathcal{G}(V)}{dV^2} = 0, \tag{51}$$

where $\epsilon = (1/\psi - 1)/(1-\gamma)$. The exact solution to this equation can be calculated numerically. The numerical values are indeed very close to those from the log-linear approximation. Theoretically, however, care must be exerted that the log-linear approximation of the aggregator f should be an increasing function of J in parameter regions of interest. Otherwise, the monotonicity axiom of preferences will be violated. (The online appendix provides more details.)

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