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Author(s): Son-Nan Chen and Stephen J. Brown
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Estimation Risk and Simple Rules for Optimal Portfolio Selection

SON-NAN CHEN* and STEPHEN J. BROWN**

I. Introduction

Elton, Gruber and Padberg (EGP) [6, 7] have recently simplified the process of constructing optimal portfolios by developing simple criteria for optimal portfolio selection which do not involve use of mathematical programming. Their simple decision rules permit one to determine easily which securities to include in an optimal portfolio and how much to invest in each. However, in practical applications of theoretical models, sample estimators are usually treated as if they were true values of unknown parameters. As a result, the effect of the standard errors of sample estimators on decision rules are completely ignored. Bawa, Brown and Klein [1] have shown that what is optimal in the absence of estimation risk is not necessarily optimal or even approximately optimal in the presence of estimation risk. Moreover, Brown [4] examined optimal portfolio choice under uncertainty for various portfolio selection procedures—the diffuse Bayes rule, the Markowitz Certainty Equivalent (CE) rule, the aggregation CE rule, and the equal weight rule,[1] and found that the diffuse Bayes rule uniformly dominates the Markowitz CE rule in repeated samples for the quadratic utility case. As the sample size increases, the Bayes rule becomes superior to the aggregation CE and the equal weight rules. In addition, the result holds even where the probability distribution of returns is seriously misspecified. Thus, Brown's [4] study has clearly indicated that, without taking estimation risk into account, portfolio selection rules other than the Bayes rule can lead investors to select suboptimal portfolios.

This paper shows by using the single index model for the return generating process that the simple decision rules for optimal portfolio selection derived by Elton, Gruber and Padberg [7] are not identical under the Bayesian and the traditional methods of analysis.[2] Moreover, in the case where short sales are not

* The University of Maryland at College Park, College Park, Maryland.

** Bell Laboratories, Murray Hill, New Jersey.

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[1] The aggregation CE rule was proposed by Blume [3]. Under this portfolio strategy individual securities are grouped into different equally weighted portfolios according to the rank of betas computed from the Sharpe single index model. Then a Certainty Equivalent rule is applied on the basis of the aggregated data. See Brown [4] for a detailed discussion of these four portfolio selection rules.

[2] The traditional method of analysis refers to the practice of considering the sample estimates as the true parameter values.

allowed the number of component securities in an optimal portfolio under the Bayesian approach can be considerably smaller than the traditional method. The results of the present study together with Brown's [3] findings demonstrate that estimation risk must be properly reflected in the process of optimal portfolio selection.

II. Single Index Model and Optimal Portfolio Construction Under Estimation Risk

Assume that the single index model is an appropriate description of the generating process of security returns. The model is defined as follows:

$$R_{it} = \alpha_i + \beta_i I_t + \varepsilon_{it}, \qquad t = 1, 2, \cdots, T \quad \text{(observations)}$$

$$I_t = \mu_M + \varepsilon_{Mt} \tag{1}$$

where R_{it} is the return on security i in period t; I_t is the return on a market index in period t; β_i is an index of systematic risk; α_i is the nonmarket return of stock i; ε_{it} is a random error term with mean zero and variance $\sigma^2_{\varepsilon i}$; μ_M is the expected return on the market index; ε_{Mt} is the random error of the market index; and σ^2_M is the variance of the market index. In addition, according to the single index model we have

$$\mathrm{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0, \qquad i \neq j, \qquad \mathrm{Cov}(\varepsilon_{it}, \varepsilon_{it'}) = 0, \qquad t \neq t',$$
$$\text{and} \quad \mathrm{Cov}(\varepsilon_{Mt}, \varepsilon_{it}) = 0.$$

The investor's objective is to find the portfolio with the highest ratio of excess return to standard deviation of return. To incorporate estimation risk into the portfolio selection process, the objective function must be expressed in terms of the parameters of the predictive distribution of security returns. Under the normality assumption of security returns, Brown [5, p. 167] has shown that the (unconditional) predictive distribution of a portfolio's return, which is assumed to follow the Sharpe's single index model, has a mean

$$\bar{R}_{p,T+1} = (\textstyle\sum_{i=1}^{N} X_i\hat{\alpha}_i) + (\sum_{i=i}^{N} X_i\hat{\beta}_i)\bar{I} \tag{2}$$

and variance

$$\sigma^2_{p,T+1} = \frac{\nu\hat{S}^2}{\nu - 2} \cdot \textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} X_iX_j\sigma_{\varepsilon ij} \cdot \left[\sigma^{*2}_M\kappa_2 + \left(1 + \frac{1}{T}\right)\right] + \sigma^{*2}_M\hat{\beta}^2 \tag{3}$$

where X_i is the portion of funds invested in security i; $\bar{I}$ is the sample mean of the market return; $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the OLS estimates of α_i and β_i; $\sigma_{\varepsilon ij}[= \mathrm{Cov}(\varepsilon_i, \varepsilon_j)]$ is the covariance between the residual return on security i and the residual return on security j;[3] $\sigma^{*2}_M[=((T - 1)(1 + 1/T)/(T - 3))\hat{\sigma}^2_M]$ is the market variance

[3] Under the assumption of the single index model the returns on securities are correlated through their common response to the movement of a market factor. Thus the residual returns (ε_{it}) of different securities are assumed to be uncorrelated. That is, $\sigma_{\varepsilon ij} = 0$ for $i \neq j$. In addition, $\sigma_{\varepsilon ij} = \sigma^2_{\varepsilon i}$ for $i = j$, the residual variance of security i.

adjusted for estimation risk; $\hat{\sigma}_M^2$ is the unbiased estimate of the market variance; $\kappa_2 = \hat{S}_\beta^2/\hat{S}^2$; $\hat{S}^2$ is the pooled residual variance;[4] $\hat{S}_\beta^2 = \hat{S}^2/[(T-1)\hat{\sigma}_M^2]$; ν denotes the degrees of freedom; and $\hat{\beta}$ [$= \sum_{i=1}^N X_i\hat{\beta}_i$] is the portfolio beta. Once the predictive distribution of the portfolio return is obtained, the objective function which reflects the effect of estimation risk is expressed as follows.[5]

$$\theta^* = \frac{\sum_{i=1}^N X_i(\bar{R}_i - R_f)}{\{K[\sum_{i=1}^N X_i^2(\sigma_{\varepsilon i}^2\hat{S}^2) + (\sigma_M^{*2}/K)(\sum_{i=1}^N X_i\hat{\beta}_i)^2]\}^{1/2}} \tag{4}$$

where $\kappa = \nu H/(\nu - 2) > 1$, $H = \sigma_M^{*2}\kappa_2 + (1 + 1/T) = (1 + 1/T)\left(\frac{T-2}{T-3}\right) > 1$,

It should be noted that the objective function is solely a function of the mean and variance. Brown [5] has illustrated that in the presence of estimation risk optimal portfolio choice should be determined by a three parameter rule. The three parameters are mean, variance and beta coefficient. The third parameter, beta coefficient, determines all higher order moments and the degree of exposure to estimation risk given the mean and variance. This implies that the criterion based on mean and variance alone is generally insufficient to rank optimal portfolios for most utility functions. However, as Brown [5] has indicated, the mean-variance criterion may provide a close approximation to the efficient set of portfolios. In the following paragraphs, the construction of optimal portfolios under the single index model will be given for two different cases—with and without short sales.

A. Estimation Risk and Optimal Portfolios When Short Sales Are Allowed

The difference between the objective function (θ^*) under estimation risk and the EGP objective function is observed. First equation (4) is rewritten as an objective function (θ) times a scalar ($K^{-1/2}$):

$$\theta^* = (1/K^{1/2})\cdot\theta \tag{5}$$

where

$$\theta = \frac{\sum_{i=1}^N X_1(\bar{R}_i - R_f)}{[\sum_{i=1}^N X_i^2(\sigma_{\varepsilon i}^2\hat{S}^2) + (\sigma_M^{*2}/K)(\sum_{i=1}^N X_i\hat{\beta}_i)^2]^{1/2}} \tag{6}$$

If we consider the terms $(\sigma_{\varepsilon i}^2\hat{S}^2)$ and (σ_M^{*2}/K) in (6) to correspond to the terms $\sigma_{\varepsilon i}^2$ and σ_M^2 as defined by EGP in their objective function, respectively, the θ in (6) is similar to the EGP objective function. Given this, the objective function

[4] For each security there are $(T-2)$ degrees of freedom associated with the residual variance. Thus there are $(T-2)N$ degrees of freedom associated with the pooled residual variance ($\hat{S}^2$). $\hat{S}^2 = \sum_{i=1}^N \sum_{t=1}^T \varepsilon_{it}^2/\nu$ and $\nu = (T-2)N$.

[5] In practice, the residual variance ($\sigma_{\varepsilon i}^2$) in (4) can be substituted by its maximum likelihood estimate without seriously affecting the predictive distribution. Brown [5] has indicated that a first order expansion of the predictive distribution around the maximum likelihood estimates of the $\sigma_{\varepsilon i}^2$ will provide a close approximation to the predictive distribution.

(θ^*) in the Bayesian case differs from the EGP objective function only by the scalar ($K^{-1/2}$). Thus, maximizing θ^* is equivalent to maximizing θ. This implies that the optimal investment proportion (X_i^*) for stock i under estimation risk can be easily obtained from the EGP optimal investment proportion by replacing EGP's $\sigma_{\varepsilon i}^2$ and σ_M^2 by $\sigma_{\varepsilon i}^2 \hat{S}^2$ and σ_M^{*2}/K, respectively. With this substitution, the optimal weight (X_i^*) for stock i in the presence of estimation risk can be expressed as:

$$X_i^* = Z_i^* / \textstyle\sum_{j=1}^N |Z_j^*|, \qquad i = 1, 2, \cdots, N \qquad (\textstyle\sum_{i=1}^N X_i^* = 1) \tag{7}$$

where

$$Z_i^* = \frac{1}{\sigma_{\varepsilon i}^2 \hat{S}^2}[(\bar{R}_i - R_f) - h(T) \cdot \hat{\beta}_i],^6 \tag{8}$$

$$h(T) = \frac{\sum_{j=1}^N \dfrac{(\bar{R}_j - R_f)\hat{\beta}_j}{\sigma_{\varepsilon j}^2 \hat{S}^2}}{K \cdot \dfrac{1}{\sigma_M^{*2}} + \sum_{j=1}^N \dfrac{\hat{\beta}_j^2}{\sigma_{\varepsilon j}^2 \hat{S}^2}} \tag{9}$$

As indicated by equations (7), (8), and (9), the Bayesian optimal portfolio becomes identical to the EGP optimal portfolio as sample size (T) approaches infinity.[7] Thus, the convergence itself does not affect optimal portfolio choices. However, equations (8) and (9) imply that estimation risk leads to a reduction in the impact of the (estimated) systematic risk ($\hat{\beta}_i$) on optimal portfolio choices.[8] The declining impact of systematic risk is consistent with the new interpretation of beta coefficient under estimation risk. That is, beta coefficient measures the degree of exposure to estimation risk (given mean and variance) (See Brown [4] for details).

To provide further insight into the difference between the Bayesian and the traditional approaches, a numerical example is shown here. The risk-return characteristics of six stocks using monthly rates of return from January 1956 to December 1978 are given in Table 1. By using the information in Table 1, the optimal weights for the Bayesian and the traditional methods of analysis are reported in Table 2.

Note that security 4 has a negative value of $(\bar{R} - R_f)/\varepsilon_{\varepsilon i}^2$. Under both approaches security 4 is held in a short position. In fact, security 4 is the only security sold short under the traditional approach. However, when admitting estimation risk

[6] $\hat{S}^2$ associated with $1/\sigma_{\varepsilon i}^2 \hat{S}^2$ will be canceled in forming the optimal weights using (7).

A recognizable form of $h(T)$ is $\left(\dfrac{\sigma_m^{*2}}{K}\right)\left[\sum_j \dfrac{(\bar{R}_j - R_f)\hat{\beta}_j}{\sigma_{\varepsilon j}^2 \hat{S}^2}\right] \Big/ \left(1 + \dfrac{\sigma_m^{*2}}{K} \cdot \sum_j \dfrac{\hat{\beta}_j^2}{\sigma_{\varepsilon j}^2 \hat{S}^2}\right)$.

[7] This is because $\lim_{T\to\infty}[\nu/(\nu - 2)] = 1$, $\lim_{T\to\infty} H = 1$, $\lim_{T\to\infty}\sigma^{*2} = \sigma_M^2$ (EGP), and the (Bayesian) $\sigma_{\varepsilon i}^2 \hat{S}^2$ converges to the EGP $\sigma_{\varepsilon i}^2$. Thus, θ^* converges to θ as $T \to \infty$. [Note that the $\hat{S}^2 \to 1$ as $T \to \infty$. See Brown [5] for details].

[8] This is because the weight associated with $\hat{\beta}_i$, $h(T)$, is an increasing function of sample size T. [K in (9) is a decreasing function of T].

Table 1

The Risk-Return Characteristics of the Six Stocks

Security	$\bar{R}_i$	$\hat{\sigma}_i^2$	$\hat{\beta}_i$	$\hat{\sigma}_{\varepsilon i}^2$	$(\bar{R}_i - R_f)/\hat{\sigma}_{\varepsilon i}^2$
1	.00937	.00866	1.175	.00638	.9201
2	.00705	.00224	.522	.00179	1.9832
3	.00944	.00603	1.160	.00381	1.5591
4	.00126	.00499	.832	.00386	−.5803
5	.01027	.00339	.925	.00197	3.4365
6	.01019	.00343	.724	.00257	2.6031

$\hat{S} = .00339, \quad \sigma_M^{*2} = .00168, \quad R_f = .0035$

Table 2

The Optimal Weights

	Security					
	1	2	3	4	5	6
Bayesian:	−.1818	.0736	−.0238	−.3481	.1664	.2063
Traditional:	.0321	.1406	.0599	−.2458	.2682	.2534

in the computation of optimal weights, security 1 and 3 which have larger beta coefficients (1.175 and 1.60) [and smaller values of $(\bar{R}_i - R_f)/\sigma_{\varepsilon i}^2$] are added in the list of short sales under the Bayesian method of analysis. Thus, securities with a higher degree of exposure to estimation risk (as measured by a larger value of beta coefficient) are ones to be held in a short position. This result is expected from equation (8). Securities with a larger beta (or higher estimation risk) tend to have greater values of $h(T)\cdot\hat{\beta}_i$ in (8) which can lead to negative weights. As a result, when estimation risk is taken into account, securities with higher estimation risk tend to be sold short. This result leads to more short sales under the Bayesian analysis.

B. Estimation Risk and Optimal Portfolios With No Short Sales

When short selling of securities is not allowed, the nonnegativity constraints must be imposed. Following the EGP results and the foregoing analysis, the Z_i^*'s when short sales are not allowed are

$$Z_i^* = \frac{1}{\sigma_{\varepsilon i}^2 \hat{S}^2} \leq [(\bar{R}_i - R_f) - \hat{\beta}_i \cdot \Phi_k], \quad i = 1, 2, \cdots, k \ (\leq N) \tag{10}$$

where

$$\Phi_k = \left[\sum_{j=1}^k \frac{\hat{\beta}_j(\bar{R}_j - R_f)}{\sigma_{\varepsilon j}^2 \hat{S}_k^2}\right] \Bigg/ \left[\frac{\nu_k H_k}{\nu_k - 2} \cdot \frac{1}{\sigma_M^{*2}} + \sum_{j=1}^k \frac{\hat{\beta}_k^2}{\sigma_{\varepsilon j}^2 \hat{S}_k^2}\right]^9 \tag{11}$$

ν_k, $\hat{S}_k^2$, and H_k are defined as in (3) with the replacement of N by k; k is the actual number of securities in the optimal portfolio. Note that Φ_k in (10) has a similar implication as the $h(T)$ in (8). The Φ_k is an increasing function of sample

[9] Φ_k in (11) can be written in a form similar to $h(T)$ in (9).

size T. As a result, with no short sales the estimated systematic risk ($\hat{\beta}_i$) has a declining impact on optimal portfolio choices under estimation risk.

The process of determining which k securities to be included in the optimal portfolio is similar to that employed by EGP. To illustrate the difference in the optimal weights between the Bayesian and the traditional methods, we use the previous six-stock data reported in Table 1. Under the Bayesian method, the optimal portfolio consists of security 6 only, which has a lower degree of exposure to estimation risk ($\hat{\beta}_6 = .72$) and a higher ratio of $(\bar{R}_i - R_f)/\sigma_{\varepsilon i}^2$ (= 2.6031).[10] The remaining five securities are rejected. However, under the traditional method of analysis, the optimal portfolio consists of five securities with their weights given as follows: $X_1 = .0217$, $X_2 = .1854$, $X_3 = .0473$, $X_5 = .3689$, and $X_6 = .3767$ (security 4 is excluded). Thus the Bayesian method leads to few securities in the optimal portfolio. This is because securities with higher estimation risk tend to have greater values of $\hat{\beta}_i \cdot \Phi_k$ in (10) which result in negative Z_i^*'s. With no short sales, those securities with a high degree of estimation risk will not be selected in the optimal portfolio. In other words, under estimation risk investors select securities which are exposed to a lower degree of estimation risk.

III. Conclusions

The effect of estimation risk on optimal portfolio choice under the framework of Elton, Gruber and Padberg [7] was examined for the single index model. Under the single index model, the criteria for optimal portfolio selection are completely different for both the traditional and the Bayesian methods of analysis. This difference reflects the added risk of an optimal portfolio due to parametric uncertainty. This increased uncertainty turns risk-adverse investors to select portfolios which minimize the overall portfolio risk including estimation risk. The results of the present study have indicated that the presence of estimation risk reduces the relative impact of estimated systematic risk on optimal portfolio choices. In addition, investors can be hurt by not taking estimation risk into account. We would conjecture that the implication of the results is beyond the present portfolio selection context.

An extension of this study should examine the effect of estimation risk on simple criteria for optimal portfolio selection under the constant correlation coefficient models. This requires a direct derivation of the predictive distribution of a portfolio's return when the correlation coefficient is unknown.

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[10] A detailed illustration of the computation is available upon request.

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